Conforming, Tetrahedral Meshing of Image Data

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with
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Motivation

• **BioMesh3D**
  - State of the start
  - Adaptive
  - Mostly good quality elements
  - May produce degenerate elements
  - Slow

• **What we’d like**
  - No degenerate elements.
  - Bounded worst-case element quality
  - Guarantees on conformity / fidelity
  - Fast
Element Quality

- Dihedral Angles
- Condition Number

![Diagram showing condition number vs. worst dihedral angle]
Approach

• Combinatoric (not variational)
  o Background mesh (structured)
  o Stencils to capture surfaces

• Inherent Tradeoff
  o Deforming background grid (small deviations)
  o Cutting / Subdividing (large deviations)
  o Labelle & Shewchuck [SIGGRAPH 2007]
Violations

• Cut-Vertex violations
  o Defined by Labelle & Shewchuck
  o Using definition $c_{12} = (1-t)v_1 + (t)v_2$
    • $c_{12}$ violates $v_1$ if $t < \alpha$
    • $c_{12}$ violates $v_2$ if $t > (1-\alpha)$
Violations

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    • $c_{12}$ violates $v_2$ if $t > (1 - \alpha)$

Guards against bad angles in the necessary in the tessellation of the remaining polygon region.
Multi-Material Challenges

- Higher dimensional features
- Interfaces may form sharp cusps
- Complexity of cases
  - Arbitrarily many cases even with linear elements
  - Not all representable
  - Snaps/warps more constrained

![Diagrams showing geometric transformations](image)
Algorithm Overview

• Begin with background tetrahedral mesh
• Compute material interface points
  o 2-material cut-points
  o 3-material triple-points
  o 4-material quadruple-points
• Generalize stencils
• Iterate over k-cells of background mesh
  o Snap/Warp violating interface points.
  o Resolve Degeneracies
• Fill in stencil tetrahedra
Representing Volumes

- Background BCC-Lattice
- Compute material transitions
Stencils

• Other stencil schemes
  o “Marching Tetrahedra”
  o “Multi-tissue Mesh Generation for Brain Images”, Liu et al. [IMR 2010]
    • Subdivision scheme
    • Different constraints
Stencils

- Face tessellations must be consistent
- Snaps can topology
- May lead to invalid meshes
Stencils

- Treat all lattice tet cases as degeneracy of 4-case
- Generalize
  1) Cutpoints always sit on primal vertices (when possible)
  2) Triplepoints always follow adjacent edge-cut movements
  3) Quadpoints always move to triple points
• **Triple-Vertex violations**
  - Natural Extension from 1D to 2D
  - Guard against bad angles
  - Polygonal region bounded by 2 lines
Violations

- Triple-Edge violations
  - Guard against bad angles
  - Polygonal region bounded by 2 lines
Additional Complexities

• Exterior Points

• Degeneracies
Graded Background

Octree Structure

Background Stencils
Evaluation
### Table 1. Torso Simulation

<table>
<thead>
<tr>
<th>Method</th>
<th>Min Angle</th>
<th>Condition</th>
<th>Iterations</th>
<th>At Threshold</th>
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<tbody>
<tr>
<td>MMStuffing</td>
<td>7.4304</td>
<td>5.42e+06</td>
<td>553</td>
<td>94.33 %</td>
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<tr>
<td>BioMesh</td>
<td>0.0000</td>
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<td>40.49 %</td>
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### Table 2. Head Simulation

<table>
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<tr>
<th>Method</th>
<th>Min Angle</th>
<th>Condition</th>
<th>Iterations</th>
<th>Mean</th>
<th>STD</th>
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<td>1.57e+18</td>
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### Table 3. Rabbit Leg Simulation

<table>
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<th>Min Angle</th>
<th>Condition</th>
<th>Iterations</th>
<th>Max E.F.</th>
<th>Max C.D.</th>
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</table>
Evaluation (Qual.)

BioMesh3D

MMStuff
Evaluation (Qual.)

BioMesh3D

MMStuff
Evaluation (Qual.)

CGAL

MMStuff
Other Domains

Multi-Ball Drop
Mesh Cutaway